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AN ERGODIC THEOREM WITH LARGE NORMALISING CONSTANTS

BY

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ABSTRACT

We generalise W. Feller's limit theorem for (independent) random variables with infinite moments.

Suppose that $\{X_n\}$ are independent identically distributed random variables, and that $E(|X_1|)=\infty$. Let $S_n = X_1 + \cdots + X_n$. In this situation, W. Feller proved

THEOREM A [3]. *If $b(n)>0$ are constants such that $b(n)/n$ increases as n, then, either* $\overline{\mathrm{Lim}}|S_n|/b(n)=\infty$ a.e. *or* $|S_n|/b(n)\to_{n\to\infty}0$ a.e.

The second alternative is characterised by $E(a(|X_1|))<\infty$ where $a=b^{-1}$. From this result, Y. S. Chow and H. Robbins deduced

THEOREM B [2]. *In the same situation, for any constants b_n either* $\overline{\mathrm{Lim}}_{n\to\infty}|S_n|/b_n=\infty$ a.e. *or* $\underline{\mathrm{Lim}}_{n\to\infty}|S_n|/b_n=0$ a.e. (*or both*).

The question arose as to whether these results remain true for a general non-integrable ergodic stationary process. In [1] we proved Theorem B for a non-integrable *positive* ergodic stationary process (theorem 1 in [1]) and showed by example that Theorem A unmodified fails in general. The example in [1] (adapted from [6]) was an ergodic, stationary process $\{X_n\}$ with $E(|X_1|)=\infty$ and $S_n/n\to 1$ a.e. The point of this note is to show that, with the minimum modification necessary to allow for this example, Theorem A is general. We prove

THEOREM A′. *Let $(X,\mathcal{B},\mu,T)$ be an ergodic measure preserving transformation ($\mu(X)=1$) and let $f:X\to\mathbf{R}$ be a measurable function. If $b(n)>0$ and $b(n)/n\uparrow\infty$ as $n\to\infty$ then either* $\overline{\mathrm{Lim}}_{n\to\infty}|\sum_{k=0}^{n-1}f\circ T^k|/b(n)=\infty$ a.e. *or* $|\sum_{k=0}^{n-1}f\circ T^k|/b(n)\to_{n\to\infty}0$ a.e.

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PROOF. Write $f_n(x) = \sum_{k=0}^{n-1} f(T^k x)$ and

$$\lambda(x) = \overline{\lim_{n\to\infty}} |f_n(x)|/b(n).$$

Then

$$\lambda(Tx) = \overline{\lim_{n\to\infty}} |f_{n+1}(x) - f(x)|/b(n)$$

$$\geqq \overline{\lim_{n\to\infty}} |f_{n+1}(x)|/b(n+1) - \lim_{n\to\infty} |f(x)|/b(n)$$

$$= \lambda(x).$$

Hence $\lambda = \lambda(x)$ is constant a.e. and we must prove that $\lambda < \infty \Rightarrow \lambda = 0$. Suppose that $\lambda < \infty$. By Egorov's theorem, we can choose $B \in \mathcal{B}$, $\mu(B) = 1/2$ such that

$$(*) \qquad \sup_{n\geqq 1} \operatorname{ess-sup}_{x\in B} |f_n(x)|/b(n) = M < \infty.$$

Let $b_1(n) = Mb(n)$, and $a(n) = b_1^{-1}(n)$. Then $a(n)/n \downarrow 0$ as $n \uparrow \infty$.

We now induce the whole process onto B, and show that $|f_n(x)|/b(n) \to 0$ a.e. on B. This will establish the theorem since λ is constant.

Let, for $x \in B$, $\phi(x) = \inf\{n \geqq 1 : T^n x \in B\}$ (the return time function of T on B) and $T_B x = T^{\phi(x)} x$ (the induced transformation of T on B). Then $(B, \mathcal{B} \cap B, \mu_B, T_B)$ is an e.m.p.t. and $\int_B \phi d\mu = 1$ (see [4] and [5] respectively).

Let $g(x) = |f_{\phi(x)}(x)|$ for $x \in B$. By construction $g(x) \leqq b_1(\phi(x))$, hence $a(g(x)) \leqq \phi(x)$ and $\int_B a(g) d\mu \leqq 1$.

Let $g_n(x) = \sum_{k=0}^{n-1} g(T_B^k x)$. The next stage in the proof is to show that

$$(**) \qquad a(g_n(x))/n \to 0 \qquad \text{a.e. on } B.$$

This is clear in case g is integrable on B, for then $g_n \sim cn$ a.e., by the Birkhoff ergodic theorem, and so

$$a(g_n(x))/n \leqq a(2cn)/n \qquad \text{for } n \text{ large}$$

$$\to 0 \qquad \text{as } n \to \infty.$$

If g is not integrable on B, we need the following

CLAIM. *$a(x) \sim C(A(x))$ as $x \uparrow \infty$ where $A(g)$ is integrable on B, $A(x)/x \downarrow$ as $x \uparrow$, $B(x) \uparrow \infty$, and $C(x)/x \downarrow 0$ as $x \uparrow \infty$.*

PROOF. Let $b(x) = a^{-1}(x)$. No generality is lost in assuming that $b(0) = 0$,

$b(n+p)=(1-p)b(n)+pb(n+1)$ for $0 \leqq p \leqq 1$, $n \in \mathbf{Z}_+$ (this assumption will not perturb $a(x)$ $(x > b(1))$ by more than 1). Sequences of this form have the property that $b(x)/x \uparrow$ as $x \uparrow$ $(x \in \mathbf{R})$ iff $b(n)/n \uparrow$ as $n \uparrow$ $(n \in \mathbf{N})$. (Differentiate)

Firstly, note that $b(n)/n \uparrow$ as $n \uparrow$ $(n \in \mathbf{N})$ means precisely that

$$b(n) = b(1) \prod_{k=2}^{n} \left(1 + \frac{\alpha_k}{k-1}\right) \qquad \text{where } \alpha_k \geqq 1.$$

Note also that

$$\frac{b(n)}{n} = b(1) \prod_{k=2}^{n} \left(1 + \frac{\alpha_k - 1}{k}\right).$$

Thus $a(x)/x \downarrow 0$ as $x \uparrow \infty \Leftrightarrow b(n)/n \to_{n\to\infty} \infty \Leftrightarrow \prod_{k=2}^{\infty}(1+(\alpha_k - 1)/k) = \infty$. Let $\mu_m = \mu(B \cap [m \leqq a(g) < m+1])$. Since $a(g)$ is integrable on B, we have that $\sum_{m=1}^{\infty}(m+1)\mu_m < \infty$.

We first find a sequence $\{D(n)\}_{n=1}^{\infty}$ satisfying

(i) $$D(n)/n \uparrow \infty \qquad \text{as } n \uparrow \infty,$$

(ii) $$\sum_{n=1}^{\infty} D(n+1)\mu_n < \infty,$$

(iii) $$b(n)/D(n) \uparrow \qquad \text{as } n \uparrow.$$

From the above remarks (about $b(n)$), $D(n)$ will need to be of the form

$$D(n) = D \prod_{k=2}^{n} \left(1 + \frac{\beta_k}{k-1}\right) \quad \text{where } \beta_k \geqq 1 \quad \text{and} \quad \prod_{k=2}^{\infty} \left(1 + \frac{\beta_k - 1}{k}\right) = \infty$$

in order to satisfy condition (i). It is easy to check that condition (iii) will be satisfied if, in addition, $\beta_k \leqq \alpha_k$.

Now choose $n_k \to \infty$ such that $\forall k \geqq 1$

(a) $\sum_{m=n_k}^{\infty}(m+1)\mu_m < 1/3^k$ and

(b) $\prod_{j=n_k+1}^{n_{k+1}}(1+(\alpha_j - 1)/j) > 2$.

Clearly, $\exists\ 1 \leqq \beta_k \leqq \alpha_k$ for $k \geqq 1$ such that

(c) $\prod_{j=n_k+1}^{n_{k+1}}(1+(\beta_j - 1)/j) = 2$.

Let $D(n) = D(1)\prod_{k=2}^{n}(1+\beta_k/(k-1))$. Conditions (i) and (iii) are satisfied. So is (ii):

$$\sum_{m=n_1}^{\infty} D(m+1)\mu_m = \sum_{k=0}^{\infty} \sum_{m=n_k}^{n_{k+1}-1} (m+1)[D(m+1)/(m+1)]\mu_m$$

$$\leqq \sum_{k=1}^{\infty} (D(n_k)/n_k)(1/3^k) \qquad \text{by (a) and (i)}$$

$$= D(1)\sum_{k=1}^{\infty}(1/3^k)\prod_{j=2}^{n_1}\left(1+\frac{\beta_j-1}{j}\right)2^k<\infty.$$

We now define $D(x)$ for $x\in\mathbf{R}$, $x\geqq 0$ by $D(0)=0$, $D(n+p)=(1-p)D(n)+pD(n+1)$, $0\leqq p\leqq 1$, $n\in\mathbf{Z}_+$ and we have that $D(x)/x\uparrow\infty$ as $x\uparrow\infty$ $(x\geqq 1)$.

It can also be checked (by differentiating) that $b(x)/D(x)\uparrow$ as $x\uparrow$ $(x>1)$.

Now let $C(x)=D^{-1}(x)$, then $C(x)\uparrow\infty$ and $C(x)/x\downarrow 0$ as $x\uparrow\infty$. Let $A(x)=D(a(x))$, then $A(x)\uparrow\infty$ as $x\uparrow\infty$ and

$$A(x)/x\downarrow \quad\text{as}\quad x\uparrow \quad \text{since } A(b(x))/b(x)=D(x)/b(x)\downarrow \quad\text{as}\quad x\uparrow.$$

Lastly $A(g)$ is integrable on B by condition (ii).

The claim is established, and we can now prove (**) in case g is not integrable on B. We have

$$A(g_n)\leqq A(g)_n\left(=\sum_{k=0}^{n-1}A(g\circ T_B^k)\right)\quad\text{because } A(n)/n\downarrow \text{ as } n\uparrow \text{ and } g\geqq 0$$

$$\sim c_1 n \quad\text{by the Birkhoff ergodic theorem, since } A(g)\in L^1(B),$$

whence

$$\begin{aligned} a(g_n)=C(A(g_n))&\leqq C(A(g)_n) &&\text{since } B(x)\uparrow \text{ as } x\uparrow\\ &\leqq C(2c_1n) &&\text{for } n \text{ large}\\ &=o(n) &&\text{as } n\to\infty. \end{aligned}$$

From (**), we deduce immediately that

$$\frac{g_n(x)}{b(n)}\to 0 \qquad \text{a.e. on } B$$

since $a(g_n)<\varepsilon n \Rightarrow g_n<b_1(\varepsilon n)<\varepsilon b_1(n)$ as $b(n)/n\uparrow$.

Now, let $\phi_n(x)=\sum_{k=0}^{n-1}\phi(T_B^k x)$. Then $T_B^k x=T^{\phi_k(x)}x$ and

$$\begin{aligned} |f_{\phi_n(x)}(x)|&=\left|\sum_{j=0}^{\phi_n(x)-1}f(T^jx)\right|\\ &=\left|\sum_{k=0}^{n-1}\sum_{j=0}^{\phi(T_B^kx)-1}f(T^jT_B^kx)\right|\\ &\leqq g_n(x). \end{aligned}$$

Hence $|f_{\phi_n(x)}(x)|/b(n)\to 0$ a.e. on B.

To finish, suppose $n=\phi_{k_n(x)}(x)+l_n(x)$ where $0\leqq l_n(x)<\phi(T_B^{k_n(x)}x)$. Then, since $\phi_n(x)\sim 2n$ a.e. on B, $k_n(x)\sim n/2$ a.e. on B, and we have

$$|f_n(x)| \leqq f_{\phi_{k_n}}(x)| + |f_{l_n(x)}(T_B^{k_n}(x))|.$$

For n large, $k_n(x) < n$ and so

$$|f_{\phi_{k_n}}(x)/b(n)| < |f_{\phi_{k_n}}(x)|/b(k_n(x)) \to 0 \qquad \text{a.e. on } B$$

and

$$|f_{l_n}(T_B^{k_n(x)}x)| \leqq b_1(l_n(x)) \qquad \text{by } (*)$$
$$\leqq b_1(\phi(T_B^{k_n}x)).$$

Since $\int_B \phi d\mu < \infty$ we have by the Borel–Cantelli lemma that

$$\phi \circ T^{k_n}/n \to 0 \qquad \text{a.e.} \qquad \text{whence}$$
$$b_1(\phi \circ T^{k_n})/b(n) \to 0.$$

This completes the proof that $\lambda = 0$. Q.E.D.

Using the methods of [2], we can now obtain

COROLLARY B′. *Let $\{X_n\}$ be an ergodic stationary process and suppose $b_n > 0$, $\overline{\text{Lim}}_{n\to\infty} b_n/n = \infty$, then, either $\overline{\text{Lim}}_{n\to\infty} |S_n|/b_n = \infty$ a.e. or $\underline{\text{Lim}}_{n\to\infty} |S_n|/b_n = 0$ a.e. (or both).*

PROOF. Following [2] verbatim, let $b(n) = n.\max\{b_k/k : 1 \leqq k \leqq n\}$. Then, $b(n) \geqq b_n$, $b(n)/n \uparrow \infty$ and $\exists n_k \to \infty$ such that $b(n_k) = b_{n_k}$.

If $\overline{\text{Lim}}_{n\to\infty} |S_n|/b_n < \infty$ on some set of positive measure, then $\overline{\text{Lim}}_{n\to\infty} |S_n|/b(n) < \infty$ on the same set, and $|S_n|/b(n) \to 0$ a.e. which implies $|S_{n_k}|/b_{n_k} \to 0$ a.e. Q.E.D.

If we were to take, in the theorem, $b(n) = n$, then our proof would establish the proposition:

$$\overline{\text{Lim}}_{n\to\infty} |S_n|/n < \infty \quad \text{a.e.} \quad \Rightarrow S_n/n \to \text{constant} \quad \text{a.e.}$$

Theorem A′ has a "dual version" for transformations preserving infinite measures:

THEOREM C. *Let $(X, \mathcal{B}, \mu, T)$ be a conservative ergodic measure preserving transformation, $\mu(X) = \infty$. Let $a(x) \uparrow \infty$, $a(x)/x \downarrow 0$ as $x \uparrow \infty$ and let $b(x) = a^{-1}(x)$. The following conditions are equivalent:*

(I) *$\exists f \in L_+^1$ such that $\underline{\text{Lim}}_{n\to\infty} S_n(f)/a(n) > 0$ on a set of positive measure,*

(II) *$\exists B \in \mathcal{B}$, $\mu(B) = 1$ such that $\int_B a(\varphi_B) d\mu < \infty$ where φ_B is the return time function of T on B,*

(III) $S_n(f)/a(n) \to \infty$ a.e. $\forall f \in L_+^1$,

(IV) $\sum_{k=0}^{n-1} \varphi_B \circ T_B^k / b(n) \to 0$ a.e. *on* B $\forall B \in \mathcal{B}$, $\mu(B) < \infty$ (*where* (*in* (I)) $S_n(f) = f + f \circ T + \cdots + f \circ T^{n-1}$).

Proof. First, note that for $f \in L_+^1$: $\underline{\mathrm{Lim}}_{n\to\infty} S_n(f)/a(n)$ is T-super invariant, and hence constant. The Hopf ergodic theorem now shows that

$$\underline{\mathrm{Lim}}_{n\to\infty} S_n(f)/a(n) = c\mu(f) \qquad \text{where } c \geqq 0 \quad \text{and} \quad \mu(f) = \int_X f d\mu.$$

Assume (I). Then $c > 0$. Fix $A \in \mathcal{B}$, $\mu(A) = 2$, and, by Egorov's theorem, choose $B \subset A$, $\mu(B) = 1$ such that

$$S_n(1_A)(x) > \delta a(n) \qquad \forall x \in B, \quad n \geqq 1.$$

Let φ be the return time function of T on B, and T_B be the transformation induced by T on B. Write $\varphi_n = \sum_{k=0}^{n-1} \varphi \circ T_B^k$. Then

$$\begin{aligned} n &\equiv S_{\varphi_n(x)}(1_B)(x) && \text{on } B \\ &\sim \frac{1}{2} S_{\varphi_n(x)}(1_A)(x) && \text{a.e. on } B \text{ by the Hopf ergodic theorem} \\ &= \frac{1}{2} \sum_{k=0}^{n-1} S_{\varphi(T_B^k x)}(1_A)(T_B^k x) \\ &> \frac{\delta}{2} \sum_{k=0}^{n-1} a(\varphi)(T_B^k x). \end{aligned}$$

Thus $\overline{\mathrm{Lim}}_{n\to\infty} (1/n) \sum_{k=0}^{n-1} a(\varphi) \circ T_B^k < \infty$ on B and so $\int_B a(\varphi) d\mu < \infty$. This is (II).

Now suppose (II); $\int_B a(\varphi) d\mu < \infty$. The proof of Theorem A′ shows that $a(\varphi_n)/n \to 0$. Since $\varphi_{S_n} \leqq n < \varphi_{S_n+1}$, we have that $a(\varphi_n)/n \to 0$ a.e. on B iff $S_n(1_B)/a(n) \to \infty$ a.e. on B which latter is the same as (III).

The above remarks show that (III) implies $a(\varphi_n)/n \to 0$ a.e. on B $\forall B \in \mathcal{B}$ (where φ is the return time function of T on B and φ_n is as above), whence, since $b(n)/n \uparrow$, $\varphi_n / b(n) \to 0$ a.e. This is (IV).

Now suppose that $\varphi_n / b(n) \to 0$ on B for some $B \in \mathcal{B}$ (where φ is the return time function of T on B and φ_n is as above). The proof of Theorem A′ shows that $\exists A \in \mathcal{B}$, $A \subseteq B$ such that if ϕ is the return time function of T_B on A, and $g = \varphi_\phi$, then $a(\sum_{k=0}^{n-1} g \,.\, T_A^k)/n \to 0$ a.e. on A. One can see easily that g is the return time function of T on A, and so $S_n(1_A)(x)/a(n) \to \infty$ a.e. on A, which is the same as (III). Q.E.D.

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